Free Choice for Weak Permission

Conditional Modality II

Dean McHugh

Linguistics Department
University of Edinburgh

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Plan

1. **The conditional reduction.** "*A* is permitted" means something like *if A were true, the deontic ideals would be met*.
2. **Strong vs. weak.** Must *every A*-world we consider be OK, or just *some*?
3. **Mixed cases** say: weak.
4. **But free choice** seems to say: strong.
5. **Alternatives** let us have both.
6. **Flattening** handles counterexamples to SDA and probability.
7. **Extensions**: other modal flavours, *any*, deontic gaps.
8. **Comparisons**: why every existing strong theory fails.

Takeaway

The slogan: permission is weak within alternatives, strong across them.

Possible worlds semantics: the basics

- A **possible world** w is a complete way things might be.
- A **proposition** is a set of worlds — the worlds where it is true.
- W is the set of all worlds; $[\![A]\!]$ is the proposition A expresses.

- Negation is complement: $[\![\neg A]\!] = \overline{[\![A]\!]}$
- Conjunction is intersection: $[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$
- Disjunction is union: $[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$

Deontic modals, the standard theory

- "You must ϕ" $= \Box\phi$;   "you may ϕ" $= \Diamond\phi$.
- Standard (Kripke/Hanson/Kratzer) semantics: fix a set of **deontically ideal worlds**, and quantify over them.

$$[\![\Box A]\!] = \{w : R(w) \subseteq [\![A]\!]\} \qquad [\![\Diamond A]\!] = \{w : R(w) \cap [\![A]\!] \neq \emptyset\}$$

- Necessity is universal, possibility existential. They are **duals**: $\Box A \equiv \neg\Diamond\neg A$.

Takeaway

On the standard theory, the modals are simply quantifiers over worlds. Today we ask: what if they are *conditionals* instead?

Conditionals via the selection functions

Following Chellas (1974, 1980), model a conditional with a **selection function**

$$f : \wp(W) \times W \to \wp(W), \qquad f(p, w) \subseteq p$$

"the worlds that result from supposing p at w."

Two conditionals, one universal and one existential:

$$[\![A \mathrel{\Box\!\!\to} C]\!] = \{w : f([\![A]\!], w) \subseteq [\![C]\!]\}$$

$$[\![A \mathrel{\Diamond\!\!\to} C]\!] = \{w : f([\![A]\!], w) \cap [\![C]\!] \neq \emptyset\}$$

Different theories fill in f differently:

- **Variably strict** (Stalnaker 1968, Lewis 1973): $f(p, w)$ = the *closest* p-worlds to w.
- **Strict** (von Fintel 2001, Gillies 2007): $f(p, w)$ = the p-worlds in w's modal horizon.

The related inferences

Free choice (Kamp 1973, Merin 1992)

$$\Diamond(A \vee B) \models \Diamond A \wedge \Diamond B$$

Alice may have an apple or a pear $\rightsquigarrow$ she may have an apple, *and* she may have a pear.

SDA: Simplification of Disjunctive Antecedents

$$(A \vee B) \mathrel{\Box\!\!\rightarrow} C \models (A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C)$$

If Alice or Bob had come, the party would have been fun $\rightsquigarrow$ if Alice had come it would have been fun, *and* if Bob had come it would have been fun.

Takeaway

These look like the same inference wearing different clothes. The conditional reduction says: they *are*.

The puzzle of free choice permission Kamp (1973) and Merin (1992)

In classical logic, $\Diamond$ is monotone: if $A \models B$ then $\Diamond A \models \Diamond B$.

Since $A \models A \vee B$, we get

$$\Diamond A \ \models \ \Diamond(A \vee B)$$

Now add free choice, $\Diamond(A \vee B) \models \Diamond A \wedge \Diamond B$. Chaining:

$$\Diamond A \ \models \ \Diamond B \qquad \textbf{for arbitrary } B$$

Permitted to have an apple $\Rightarrow$ permitted to commit arson. This is the **free choice paradox**. Something must give.

Your turn

Before we go on, record your own judgements. Say “true”, “false”, or “it depends” for each.

1. *There are two switches, A and B. The rules say A must be up; B may be in any position.*
   Are the switches allowed to be in the same position?
2. *Angelica must not eat soy, gluten, fish, or nuts.*
   Is Angelica allowed to eat food?
3. *John’s doctor permits him at most seven cigarettes a day.*
   Is John allowed to smoke more than six cigarettes a day?

Hold onto your answers. We come back to them in Part 2.

The idea

> *"It is analytic of the notion of obligation that if an obligation is not fulfilled, then something has gone wrong."*
>
> — *Anderson (1967:346–47)*

So: *You must not murder* means *if you murdered, the rules would be broken.*

$$\Box A \;\equiv\; \neg A \mathrel{\Box\!\!\rightarrow} \neg \textsc{ok}$$

And permission? Two options — this is the whole dispute:

Strong: $\Diamond A \equiv A \mathrel{\Box\!\!\rightarrow} \textsc{ok}$ **Weak:** $\Diamond A \equiv A \mathrel{\Diamond\!\!\rightarrow} \textsc{ok}$

Advocates of **strong**: von Wright (1968), Hilpinen (1982), Lokhorst (1997), Asher & Bonevac (2005), Barker (2010), Li (2025). Today's argument: they are all wrong.

The model, formally

A model $M = \langle W, f, V, \textsc{ok} \rangle$ where

W a non-empty set of possible worlds

f $\wp(W) \times W \to \wp(W)$, a conditional selection function, $f(p, w) \subseteq p$

V $\mathsf{At} \to \wp(W)$, a valuation

$\textsc{ok}$ $W \to \wp(W)$: $\textsc{ok}_w$ = the worlds where w's deontic ideals are met

Language: $A ::= p \mid \neg A \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A \mid A \mathrel{\Box\!\!\to} B \mid A \mathrel{\Diamond\!\!\to} B$

$$[\![p]\!] = V(p)$$
$$[\![\neg A]\!] = \overline{[\![A]\!]}$$
$$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$$
$$[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$$

$$[\![\Box A]\!] = \{w : f([\![\neg A]\!], w) \subseteq \textsc{ok}_w\}$$
$$[\![A \mathrel{\Diamond\!\!\to} C]\!] = \{w : f([\![A]\!], w) \cap [\![C]\!] \neq \emptyset\}$$
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The two candidate clauses for possibility

Strong

$$[\![\Diamond A]\!] = \{w : f([\![A]\!], w) \subseteq \textsc{ok}_w\}$$

A is permitted iff every selected *A*-world is OK.

Weak

$$[\![\Diamond A]\!] = \{w : f([\![A]\!], w) \cap \textsc{ok}_w \neq \emptyset\}$$

A is permitted iff some selected *A*-world is OK.

Takeaway

Strong: supposing *A* must *guarantee* that things are OK.
Weak: supposing *A* need only be *compatible* with things being OK.

Two housekeeping remarks

1. Which conditional? The subjunctive.

- A security guard required to stay at his post can say *I'm not allowed to be elsewhere right now*.
- Subjunctive reading: *if I were not at my post, the rules would be broken* — fine.
- Indicative reading: *if I am not at my post, the rules are broken* — odd, because indicatives are restricted to worlds compatible with the common ground (Stalnaker 1975), and we all know he *is* at his post.

2. What is OK?

- Simplest model: each world w sorts worlds into $\textsc{ok}_w$ and $\overline{\textsc{ok}_w}$.
- Compatible with Kratzer (1977): let $\textsc{ok}_w = \textsc{best}(b, g, w)$, the modal-base worlds that come closest to satisfying the ordering source.
- Nothing today turns on the choice.

Worked derivation: does the strong theory validate free choice?

Suppose $\Diamond(A \vee B)$ on the **strong** analysis:

$$f(\llbracket A \rrbracket \cup \llbracket B \rrbracket, w) \subseteq \textsc{ok}_w \tag{1}$$

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Now assume **SDA** holds for $\Box\!\!\rightarrow$. Written with selection functions, SDA amounts to:

$$f(p \cup q, w) \subseteq X \implies f(p, w) \subseteq X \text{ and } f(q, w) \subseteq X \tag{2}$$

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From (1) and (2):

$$f(\llbracket A \rrbracket, w) \subseteq \textsc{ok}_w \quad \text{and} \quad f(\llbracket B \rrbracket, w) \subseteq \textsc{ok}_w$$

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$$f(\llbracket A \rrbracket, w) \subseteq \text{OK}_w \quad \text{and} \quad f(\llbracket B \rrbracket, w) \subseteq \text{OK}_w$$

which is exactly $\Diamond A \wedge \Diamond B$ on the strong analysis. □

Takeaway

Strong permission *reduces free choice to SDA*. That is the tradition's big selling point — and what we now have to pay for.

Why the weak theory can't just copy this

The weak analysis needs the *existential* analogue of SDA:

$$(A \vee B) \Diamond\!\!\rightarrow C \;\models\; (A \Diamond\!\!\rightarrow C) \wedge (B \Diamond\!\!\rightarrow C)$$

But on essentially every conditional plugged into the reduction, the strong form validates SDA while its weak dual does not (Asher & Bonevac 2005:306):

- strict conditional (von Wright 1968, Li 2025)
- variably strict conditional (Hilpinen 1982, Nute 1985)
- relevance conditional (McArthur 1981, Mares 1992)
- conditional of non-monotonic logic (Asher & Bonevac 2005)
- conditional of linear logic (Barker 2010)

This is why the field went strong. Part 4 shows how to get out.

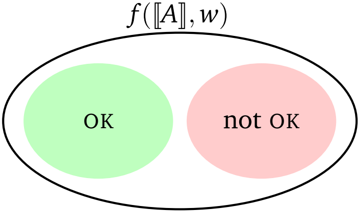
Part 2

Mixed cases

The test

The strong analysis says the permission statement and the conditional are equivalent. So put them side by side.

A **mixed case** is one where supposing "if A" gives you *some* worlds where the ideals are met and *some* where they are violated.



Weak says: permitt

Strong says: not pe

Which do speakers agree with? Let's look.

Mixed case 1: the switches

There are two switches, A and B. Each can be up or down. The rules specify that A must be up and that B may be in any position.

(a) Are the positions of the switches allowed to agree?

(b) If the positions of the switches agreed, would the rules be met?

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(b) If the positions of the switches agreed, would the rules be met?

(a) Yes. (b) It depends.

Why mixed? $\uparrow\uparrow$ agrees and is OK; $\downarrow\downarrow$ agrees and is not OK. Both are selected.

Mixed cases 2–4

Angelica

Angelica has many allergies. She must not eat soy, gluten, fish, or nuts.

(a) Is Angelica allowed to eat food?

(b) If Angelica ate food, would her dietary rules be met?

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Alice and Bob in Quebec

Alice and Bob know English and French and talk in English. A new rule requires all employees to speak French at work. They keep speaking English.

(a) Does the rule permit them to talk to each other at work?

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John's cigarettes (Fox 2007b:103)

John's doctor has permitted him to smoke at most seven cigarettes per day.

(a) Is John allowed to smoke more than six a day?

(b) If he smoked more than six, would the doctor's rules be met?

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Naturally occurring examples

Am I allowed to have help when I am voting? Yes! In many forms.
— *Maine Students Vote (2025)*

The injured party is allowed to be present during investigative acts in the phase of a judicial investigation. Specifically, he is allowed to be present at [...] the questioning of a witness, but only if it is expected that the witness will not come to the trial.
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Compare the corresponding conditionals, which cannot be asserted here:

(a) If I have help when I am voting, the rules will be met.

(b) If the injured party is present during investigative acts ..., the rules will be met.

The argument, in one line

In mixed cases, permission statements pattern with the weak conditional, not the strong one.

	"*A* **is permitted**"	"**If** *A*, **OK**"
Switches agree	Yes	It depends
Angelica eats food	Yes	It depends
Alice & Bob talk	Yes	It depends
John smokes >6	Yes	It depends

Takeaway

If you want a conditional analysis of permission, the conditional should be weak: *A* is permitted iff supposing *A* returns *some* OK world.

A complication: strong inferences

Even though the answer to each question is “Yes”, asserting these *out of the blue* feels misleading:

(a) The switches are allowed to agree.

(b) Alice and Bob are allowed to talk to each other at work.

(c) John has permission to smoke more than six cigarettes per day.

They suggest, respectively:

(a′) They’re allowed to be both up *and* allowed to be both down.

(b′) They’re allowed to talk in *any* language.

(c′) John may smoke seven, eight, nine, . . .

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If permission were strong, these would be entailments. Isn’t that evidence for strong after all?

No: three diagnostics say these are pragmatic

1. Sensitive to the question under discussion

- Answering the polar question *Is A permitted?* $\rightarrow$ "Yes" is fine.
- Answering *What positions are allowed?* with "They're allowed to agree" is misleading.
- QUD-sensitivity is a hallmark of pragmatic inference (Welker 1994, von Kuppevelt 1996, Breheny et al. 2006, Zondervan 2009, Degen & Tanenhaus 2015, Ronai & Xiang 2021).

No: three diagnostics say these are pragmatic

2. Cancellable.

- *The switches are allowed to be in the same position, but they're not allowed to be both down.* Fine.
- *Any configuration where they agree is allowed, but they're not allowed to both be down.* Contradictory.

No: three diagnostics say these are pragmatic

3. Metalinguistic negation contour.

- *Max didn't talk to Anna OR Robin, he talked to BOTH.*
- *Arseny doesn't LIKE mushrooms, he LOVES them.*
- *I didn't trap two monGEESE, I trapped two monGOOSES.* (Horn 1985)
- *Alice is not allowed to paint the house RED, she is allowed to paint it SCARLET.*

Your turn

Construct your own mixed case.

1. Pick a rule system (a game, a workplace policy, a dietary restriction).
2. Find a description A such that some ways of making A true comply and some don't.
3. Ask a partner: (a) Is A allowed? (b) If A, would the rules be met?
4. Do you get the contrast?

Follow-up: can you find a case where the contrast *doesn't* show up? What is different about it? (We return to this in Part 7 — Kamp's and Prior's "coarseness" cases.)

Part 3

Free choice bites back

Prior saw it in 1959

> *[W]e would not ordinarily infer "It is permitted either to smoke or to embezzle" from "It is permitted to smoke." [. . .] but α (say my feeding this man) is usually considered permissible even if some ways of doing α (e.g. feeding him with poison) are not, so long as some are; and smoking is one way of smoking-or-embezzling.*
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Prior's tension.

- Permission looks **weak/existential** (some ways suffice).
- But "*A* is permitted" does *not* imply "*A* or *B* is permitted".

These pull in opposite directions.

The problem for weak permission

Suppose "if you smoked or embezzled". Plausibly, in at least one selected world, you smoke and don't embezzle.

That world is OK. So some selected $(A \vee B)$-world is OK.

Weak permission therefore predicts *You may smoke or embezzle* to be true. Wrong.

General form

Whenever *some* selected $A \vee B$-world is OK, the weak analysis predicts $\Diamond(A \vee B)$, even if *no* B-world is OK.

The strong analysis avoids this: to be permitted to smoke-or-embezzle, *all* selected such worlds must be OK, including the embezzling ones.

So why not just keep the strong theory?

Because it has counterexamples of its own.

The overarching issue: disjunction is special.

Two *necessarily equivalent* sentences can behave differently under permission, where the only relevant difference is that one is disjunctive and the other is not.

Problem 1: the tickets

A concert venue limits each customer to at most five tickets.

(8) Alice is allowed to buy more than four tickets. True

a. $\not\rightsquigarrow$ Alice is allowed to buy five tickets.

b. $\not\rightsquigarrow$ Alice is allowed to buy more than five tickets.

(9) Alice is allowed to buy five or more tickets. False

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But *buys more than four* $\Leftrightarrow$ *buys five or more*. Necessarily equivalent!

Takeaway

This breaks any strong theory whose conditional validates **Substitution** (LLE/RCEA): if A and B are true at the same worlds, $A \mathrel{\Box\!\!\rightarrow} C$ and $B \mathrel{\Box\!\!\rightarrow} C$ have the same truth value.
That includes the strict conditional, the variably strict conditional, and the non-monotonic conditional.

Problem 2: De Morgan and duality

(10a) Out of *Beloved* and *Song of Solomon*, you are required to read *Beloved*, but you are not required to read both.

$$\Box A \wedge \neg\Box(A \wedge B)$$ Fine

(10b) Out of *Beloved* and *Song of Solomon*, you are required to read *Beloved*, but you are permitted to skip either of them.

$$\Box A \wedge \Diamond(\neg A \vee \neg B)$$ Odd

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Intuitively they are not equivalent:

(11) You are not required to read both. $\neg\Box(A \wedge B)$

a. $\not\rightsquigarrow$ You are not required to read A.
b. $\not\rightsquigarrow$ You are not required to read B.

(12) You are permitted to skip either. $\Diamond(\neg A \vee \neg B)$

a. $\rightsquigarrow$ You are permitted to skip A.
b. $\rightsquigarrow$ You are permitted to skip B.

This is a problem for uniform SDA/free-choice accounts like Willer (2018) and Li (2025). Cf. Ciardelli, Zhang & Champollion (2018) on epistemics.

Where we are at the halfway mark

	Strong	**Weak**
Mixed cases (§3)	✗	✓
Free choice via SDA	✓	✗
Tickets (Substitution)	✗	—
De Morgan / duality	✗	—

We want the weak column's first row *and* the strong column's second.

Enter alternatives.

— *five minute break* —

Part 4

Alternatives

The core move

Standard semantics: a sentence denotes one proposition.

Alternative semantics (Hamblin 1973; here following Alonso-Ovalle 2006): a sentence denotes a set of propositions — its **alternatives**.

And: a conditional is true iff, for each alternative of the antecedent, supposing it delivers the consequent.

Also: Alonso-Ovalle (2009), Ciardelli (2016), Santorio (2018), Willer (2018), Cariani & Goldstein (2020), Khoo (2022).

Takeaway

Disjunction is the paradigm alternative-generator: *A or B* contributes *two* propositions, not their union.

Notation and truth

- $[\![A]\!]^w$ = A's semantic value at w: a *set of propositions*.
- A is **true** at w iff some alternative is true there: $w \in \bigcup [\![A]\!]^w$.
- Abbreviate: $|A|^w := \bigcup [\![A]\!]^w$. Drop superscripts when safe.
- Entailment as usual: $A \models C$ iff C is true wherever A is.

Clauses

$$
\begin{aligned}
[\![p]\!] &= \{V(p)\} \\
[\![\neg A]\!] &= \{\overline{\bigcup [\![A]\!]}\} \\
[\![A \wedge B]\!] &= \{p \cap q : p \in [\![A]\!], q \in [\![B]\!]\} \\
[\![A \vee B]\!] &= [\![A]\!] \cup [\![B]\!]
\end{aligned}
$$

Note: negation always yields a *single* alternative. This will do a lot of work later.

The flattening operator

$$[\![!A]\!] = \left\{ \bigcup [\![A]\!] \right\}$$

Also called *existential closure*, sometimes written $\exists$.

It collapses a set of alternatives into a single one, leaving truth conditions unchanged.

$[\![p \vee q]\!]$	$= \{\|p\|,\ \|q\|\}$	two alternatives
$[\![!(p \vee q)]\!]$	$= \{\|p\| \cup \|q\|\}$	one alternative

Same truth conditions. Different alternatives. That difference is visible to conditionals and modals. Part 5 puts it to work.

Alonso-Ovalle's conditionals

$$[\![A \mathrel{\Box\!\!\rightarrow} C]\!] = \big\{\{w' : \forall p \in [\![A]\!],\ f(p, w') \subseteq |C|\}\big\}$$

$$[\![A \mathrel{\Diamond\!\!\rightarrow} C]\!] = \big\{\{w' : \forall p \in [\![A]\!],\ f(p, w') \cap |C| \neq \emptyset\}\big\}$$

Universal quantification over alternatives, in both cases.

These validate SDA for both conditionals:

$$(A \vee B) \mathrel{\Box\!\!\rightarrow} C \models (A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C)$$

$$(A \vee B) \mathrel{\Diamond\!\!\rightarrow} C \models (A \mathrel{\Diamond\!\!\rightarrow} C) \wedge (B \mathrel{\Diamond\!\!\rightarrow} C)$$

Why believe SDA for $\Diamond\!\!\rightarrow$? *If the die landed 1 or 2, you could win* has a reading implying both *if it landed 1, you could win* and *if it landed 2, you could win* (Alonso-Ovalle 2006:18–19).

Worked derivation: SDA falls out

Show: $(A \vee B) \Diamond\!\!\rightarrow C \models (A \Diamond\!\!\rightarrow C) \wedge (B \Diamond\!\!\rightarrow C)$.

Suppose $(A \vee B) \Diamond\!\!\rightarrow C$ is true at w. By the clause:

$$\forall p \in [\![A \vee B]\!] : f(p, w) \cap |C| \neq \emptyset$$

Worked derivation: SDA falls out

Show: $(A \vee B) \mathbin{\Diamond\!\!\rightarrow} C \models (A \mathbin{\Diamond\!\!\rightarrow} C) \wedge (B \mathbin{\Diamond\!\!\rightarrow} C)$.

Suppose $(A \vee B) \mathbin{\Diamond\!\!\rightarrow} C$ is true at w. By the clause:

$$\forall p \in [\![A \vee B]\!] : \ f(p, w) \cap |C| \neq \emptyset$$

But $[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$. So:

$$\forall p \in [\![A]\!] : f(p, w) \cap |C| \neq \emptyset \quad \text{and} \quad \forall q \in [\![B]\!] : f(q, w) \cap |C| \neq \emptyset$$

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Which are precisely the truth conditions of $A \mathrel{\Diamond\!\!\rightarrow} C$ and $B \mathrel{\Diamond\!\!\rightarrow} C$. $\square$

Takeaway

SDA is now *trivial*: it holds because $[\![A \vee B]\!]$ literally *contains* $[\![A]\!]$'s and $[\![B]\!]$'s alternatives, and we quantify universally over them. No constraint on f needed.

Wrinkle: downward-entailing environments

Alonso-Ovalle's semantics predicts a *weak* reading under negation:
$\neg((A \vee B) \mathrel{\Box\!\!\rightarrow} C) \equiv \neg(A \mathrel{\Box\!\!\rightarrow} C) \vee \neg(B \mathrel{\Box\!\!\rightarrow} C)$.

Santorio (2018:524) says that's wrong:

(17a) I doubt that, if Alice or Bob went, the party would be fun.

(17b) No girl would be happy if she failed algebra or geography.

(17a) implies the speaker doubts *both* conditionals; (17b) implies *no* girl would be happy if she failed algebra *and* none if she failed geography.

Conditionals keep their strength under negation:
$\neg((A \vee B) \mathrel{\Box\!\!\rightarrow} C) \equiv \neg(A \mathrel{\Box\!\!\rightarrow} C) \wedge \neg(B \mathrel{\Box\!\!\rightarrow} C)$.

Fix: homogeneity presuppositions

Following Santorio (2018) and Cariani & Goldstein (2020): all alternatives must have the *same status* with respect to the consequent.

(18) Universal conditional

a. $[\![A \mathrel{\Box\!\!\rightarrow} C]\!]$ is defined only if $\forall p, q \in [\![A]\!]$: $f(p,w) \subseteq |C|$ iff $f(q,w) \subseteq |C|$
b. if defined, $[\![A \mathrel{\Box\!\!\rightarrow} C]\!] = \{\{w' : \forall p \in [\![A]\!], f(p,w') \subseteq |C|\}\}$

(19) Existential conditional

a. $[\![A \mathrel{\Diamond\!\!\rightarrow} C]\!]$ is defined only if $\forall p, q \in [\![A]\!]$: $f(p,w) \cap |C| \neq \emptyset$ iff $f(q,w) \cap |C| \neq \emptyset$
b. if defined, $[\![A \mathrel{\Diamond\!\!\rightarrow} C]\!] = \{\{w' : \forall p \in [\![A]\!], f(p,w') \cap |C| \neq \emptyset\}\}$

All or none. This is the same move made for plural predication and definite descriptions.

Now extend to modals — for free

The conditional reduction is a recipe: give me a semantics for conditionals, I return a semantics for modals.

$$\Box A \equiv \neg A \mathrel{\Box\!\!\rightarrow} \neg \textsc{ok} \qquad \Diamond A \equiv A \mathrel{\Diamond\!\!\rightarrow} \textsc{ok}$$

Plugging in (18) and (19):

(20) Obligation

a. $[\![\Box A]\!]$ defined only if $\forall p, q \in [\![\neg A]\!]$: $f(p,w) \subseteq \textsc{ok}_w$ iff $f(q,w) \subseteq \textsc{ok}_w$
b. $[\![\Box A]\!] = \{\{w' : \forall p \in [\![\neg A]\!],\ f(p,w') \subseteq \textsc{ok}_{w'}\}\}$

(21) Permission

a. $[\![\Diamond A]\!]$ defined only if $\forall p, q \in [\![A]\!]$: $f(p,w) \cap \textsc{ok}_w \neq \emptyset$ iff $f(q,w) \cap \textsc{ok}_w \neq \emptyset$
b. $[\![\Diamond A]\!] = \{\{w' : \forall p \in [\![A]\!],\ f(p,w') \cap \textsc{ok}_{w'} \neq \emptyset\}\}$

Worked derivation: free choice

Show: $\Diamond(A \vee B) \models \Diamond A \wedge \Diamond B$.

Suppose $\Diamond(A \vee B)$ true at w. By (21b):

$$\forall p \in [\![A \vee B]\!] : f(p, w) \cap \textsc{ok}_w \neq \emptyset$$

Worked derivation: free choice

Show: $\Diamond(A \vee B) \models \Diamond A \wedge \Diamond B$.

Suppose $\Diamond(A \vee B)$ true at w. By (21b):

$$\forall p \in [\![A \vee B]\!] : \ f(p, w) \cap \text{OK}_w \neq \emptyset$$

Since $[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$:

$$\forall p \in [\![A]\!] : f(p, w) \cap \text{OK}_w \neq \emptyset \quad \text{and} \quad \forall q \in [\![B]\!] : f(q, w) \cap \text{OK}_w \neq \emptyset$$

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$$\forall p \in [\![A]\!] : f(p, w) \cap \textsc{ok}_w \neq \emptyset \quad \text{and} \quad \forall q \in [\![B]\!] : f(q, w) \cap \textsc{ok}_w \neq \emptyset$$

i.e. $\Diamond A$ and $\Diamond B$. $\square$

And dual prohibition comes free too

$\neg\Diamond(A \vee B) \models \neg\Diamond A \wedge \neg\Diamond B$.
Alice may not have an apple or a pear $\rightsquigarrow$ she may not have an apple, and she may not have a pear.

This falls out of the homogeneity presupposition in (21a): if the alternatives must be all-or-none, negating "all" gives you "none".

Why this is an improvement

- **Over Zimmermann (2000), Geurts (2005), Aloni (2007), Simons (2005):** they derive a *weak* reading for $\neg\Diamond(A \vee B)$, predicting $\neg\Diamond A \vee \neg\Diamond B$. We get the strong one.
- **Over Goldstein (2019):** he stipulates homogeneity. We derive it from independently attested behaviour of conditionals.

Takeaway

Prior's tension dissolves. Permission is weak (existential) within each alternative, but strong (universal) across alternatives.

Back to the tickets

- *Alice buys more than four tickets* — not disjunctive, so one alternative.
  Among the selected worlds is one where she buys exactly five.
  That world is OK. So (8) is true. ✓

- *Alice buys five or more tickets* — disjunctive, so multiple alternatives.
  Every alternative must be permitted. No world where she buys more than five is OK. So (9) is false. ✓

Same truth conditions. Different alternatives. Different predictions.
Exactly what the data demanded.

Back to De Morgan

$\neg(A \wedge B)$ has a single alternative (negation flattens).

So $\neg\Box(A \wedge B)$ is true iff *some* selected $\neg(A \wedge B)$-world is OK.

$\Rightarrow$ (10a) $\Box A \wedge \neg\Box(A \wedge B)$ is consistent. ✓

$\Diamond(\neg A \vee \neg B)$ is disjunctive, so entails $\Diamond\neg A$.

$\Rightarrow$ (10b) $\Box A \wedge \Diamond(\neg A \vee \neg B)$ is contradictory. ✓

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And obligation behaves

Since negations have one alternative, alternatives are inert for obligation. The theory of $\Box$ collapses to the alternative-free version of §2.

Bobby must clean the kitchen or his bedroom is true iff, were he to do neither, the rules would be broken.
And crucially we do *not* predict $\Box(A \vee B) \models \Box A \wedge \Box B$.

Your turn

Compute the alternatives.

For each, give $[\![\cdot]\!]$ as a set of propositions, and say how many alternatives there are.

1. $p \vee q$
2. $\neg(p \vee q)$
3. $\neg p \wedge \neg q$
4. $(p \vee q) \wedge r$
5. $!(p \vee q) \wedge r$
6. $\neg\neg(p \vee q)$

Then: which of $\Diamond$ applied to each yields a free choice inference? (Item 6 is a trap — we return to it in Part 6.)

Part 5

Ambiguity via flattening: counterexamples to SDA, and probability

A worry about semantic free choice

Our account makes free choice a semantic entailment:
$\Diamond(A \vee B) \models \Diamond A \wedge \Diamond B$.

Semantic accounts have to answer for embedded environments. Negation we've handled. But consider **probability operators**.

> *It's time for dessert. Alice's parents roll a die. If it lands 1 or 2, she is allowed fruit (and only fruit). If 3 or 4, ice cream (and only ice cream). If 5 or 6, cake (and only cake). The die has been cast; we don't know how it landed.*
>
> *What is the probability that Alice is allowed to have ice cream or cake?*

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>
> *What is the probability that Alice is allowed to have ice cream or cake?*

Easily accessible answer: two thirds.

Why that's a problem

Probability of *allowed ice cream* ***and*** *allowed cake*? Zero — however the die lands, she gets at most one dessert.

But **entailment preserves probability** (Kolmogorov 1933): if $A \models B$ then $\Pr(A) \leq \Pr(B)$.

If $\Diamond(A \vee B) \models \Diamond A \wedge \Diamond B$, then

$$\Pr\big(\Diamond(A \vee B)\big) \;\leq\; \Pr(\Diamond A \wedge \Diamond B)$$

But we judged $\frac{2}{3} \leq 0$. Contradiction.

Good news: exactly this problem arose for disjunctive antecedents in the conditionals literature, and was solved there. The conditional reduction lets us import the solution.

The ambiguity approach

Take *A or B* to be ambiguous between

$$A \vee B \qquad \text{and} \qquad !(A \vee B)$$

- $A \vee B$ validates SDA (multiple alternatives).
- $!(A \vee B)$ validates it only if $f(p, w) \cup f(q, w) \subseteq f(p \cup q, w)$ — generally, it doesn't.

Ambiguity endorsed by: Loewer (1976), McKay & van Inwagen (1977), Warmbrōd (1981), Hilpinen (1982), Pollock (1984), Alonso-Ovalle (2009), Santorio (2018), Khoo (2021b), Cariani & Goldstein (2020).

Two independent motivations follow.

Motivation 1: McKay & van Inwagen

(23) If WWII Spain had joined the Allies or the Axis, they would have joined the Axis. Sounds true

Under unrestricted SDA this would imply *if Spain had joined the Allies, they would have joined the Axis* — absurd.

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Nute's (1980a) contrast:

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Same antecedent. Opposite verdicts. Ambiguity explains it:

(23) $= \,!(A \vee B) \;\Box\!\!\rightarrow B$	no SDA
(24) $= (A \vee B) \;\Box\!\!\rightarrow C$	SDA $\Rightarrow$ predicts falsity

Why a non-ambiguity account struggles

Without ambiguity, the two antecedents mean the same thing.

It is widely assumed that two counterfactual antecedents with the same meaning, in the same context, have the same **counterfactual domain**.

So (23) and (24) share a domain D.

- For (23) to be true, Spain joins the Axis at every $w \in D$.
- At every such w, Hitler is pleased.
- So (24) is true too. Wrong result.

This also casts doubt on pragmatic-context-shift accounts, e.g. Fine's (2012:232) "Suppositional Accommodation", which concerns only whether the antecedent is counterfactually possible — so it can't predict that changing the *consequent* affects SDA.

Motivation 2: conditionals under *probably*

Alice threw a six-sided die. Bob bet the outcome would be 3 or 4. Mary says it landed 2, 3, or 4; Sue says 3, 4, or 5. We have no idea whether the reports are accurate. *(Santorio 2021:633)*

(25) If Mary's report is accurate or Sue's report is accurate, probably Bob won.
Has a true reading

- $(A \vee B) \mathrel{\Box\!\!\rightarrow}$ probably C: *if Mary's is accurate, probably Bob won* $(\frac{2}{3})$ *and if Sue's is accurate, probably Bob won* $(\frac{2}{3})$. True.
- $!(A \vee B) \mathrel{\Box\!\!\rightarrow}$ probably C: *if the die landed between 2 and 5, probably Bob won* — only $\frac{1}{2}$. False.

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- $!(A \vee B) \mathrel{\Box\!\!\rightarrow}$ probably C: *if the die landed between 2 and 5, probably Bob won* — only $\frac{1}{2}$. False.

(26) If the die landed 2, 3, or 4, probably Bob won. Has a true reading

- Now reversed: !-reading true; unflattened reading false (if it landed 2, Bob certainly lost).

We need both readings.

When does each reading appear?

An ambiguity account is only predictive with a generalisation about availability. Proposed:

(27)

If A or B, C may be parsed as $!(A \vee B) \mathrel{\Box\!\!\rightarrow} C$ when C entails $A \vee B$ or contains a probability operator.

Check:

- (23) … *they would have joined the Axis*: consequent entails the antecedent. ✓ Flattening available.
- (24) … *Hitler would have been pleased*: neither condition. ✗ No flattening $\Rightarrow$ SDA $\Rightarrow$ false. ✓
- (25), (26): probability operator. ✓ Both readings available.

Whether (27) follows from deeper principles is open.

Permission under probability, solved

Back to the dessert case.

- **Two thirds** answer = the flattened reading, $\Diamond!(A \vee B)$.
  No free choice inference; just "the die landed 3–6".

- **Zero (or undefined)** answer = the unflattened reading, $\Diamond(A \vee B)$, which entails $\Diamond A \wedge \Diamond B$.
  False (or presupposition failure) however the die lands.

Takeaway

The ambiguity approach licenses both intuitive answers. Semantic free choice survives the probability test.

Worry: could a McKay–van Inwagen example flatten $\Diamond(A \vee B)$ to $\Diamond!(A \vee B)$, killing free choice in unembedded contexts? No: by (27) we'd need $\textsc{ok}_w \subseteq |A|$ or $\textsc{ok}_w \subseteq |B|$. Since the valuation and $\textsc{ok}_w$ are separate model components, we can always find a model where these fail.

Your turn

Flatten or not?

For each, say which reading generalisation (27) makes available, and what it predicts.

1. If Ann or Bob had come, Bob would have come.
2. If Ann or Bob had come, the party would have been fun.
3. If Ann or Bob had come, it's likely the party would have been fun.
4. You may take the bus or the train.
5. What's the probability that you may take the bus or the train?

Harder: design a scenario that would be a genuine counterexample to (27). What would it have to look like?

Part 6

Extensions

Free choice isn't just deontic

(28a) Mr. X might be in Victoria or in Brixton. (Zimmermann 2000:ex. 6)

(28b) I can lick any man in the house, or drink the lot of you under the table. (Lewis 1977:ex. 4)

(28c) It is legal for you to report this as taxable income or for me to claim you as a dependent. (Lewis 1977:ex. 5)

Epistemic, ability, legal. In each, $\Diamond(A \vee B) \rightsquigarrow \Diamond A \wedge \Diamond B$.

Can the conditional reduction generalise?

Conditionalising the standard theory

Introduce $\Box_0, \Diamond_0, \Box_1, \Diamond_1, \ldots$ with accessibility relations $R_0, R_1, \ldots$ and the standard clauses

$$[\![\Box_i A]\!]^s = \{w' : R_i(w') \subseteq [\![A]\!]^s\} \qquad [\![\Diamond_i A]\!]^s = \{w' : R_i(w') \cap [\![A]\!]^s \neq \emptyset\}$$

Now define, for each R_i:

$$f_i(p, w) = R_i(w) \cap p \qquad \textsc{ok}_{i,w} = R_i(w)$$

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Now define, for each R_i:

$$f_i(p, w) = R_i(w) \cap p \qquad \textsc{ok}_{i,w} = R_i(w)$$

Then the two theories collapse:

$$\{w' : R_i(w') \subseteq [\![A]\!]^s\} = \{w' : f_i([\![\neg A]\!]^s, w') \subseteq \textsc{ok}_{i,w'}\}$$
$$\{w' : R_i(w') \cap [\![A]\!]^s \neq \emptyset\} = \{w' : f_i([\![A]\!]^s, w') \cap \textsc{ok}_{i,w'} \neq \emptyset\}$$

But only for the alternative-free version. Unlike the standard theory, the conditional reduction integrates with alternatives: just swap f for f_i in (20)–(21). Result: free choice for arbitrary modal flavours.

First order: free choice *any*

Vendler (1967:79–80): *Take any one of them* “grants you the unrestricted liberty of individual choice”.

(31) a. You may take any card.
     b. ⇒ For each card, you may take it.

(32) a. If you took any card, the rules would be met.
     b. ⇒ For each card, if you took it, the rules would be met.

Assuming *any* is existential (Horn 1972, Ladusaw 1979):

Propositional	First order
$\Diamond(A \vee B) \Rightarrow \Diamond A \wedge \Diamond B$	$\Diamond\exists x Ax \Rightarrow \forall x \Diamond Ax$
$(A \vee B) \mathrel{\Box\!\!\rightarrow} C \Rightarrow (A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C)$	$(\exists x Ax) \mathrel{\Box\!\!\rightarrow} C \Rightarrow \forall x(Ax \mathrel{\Box\!\!\rightarrow} C)$

The parallel between SDA and free choice extends into the first-order domain.

First-order alternative semantics

Models $M = \langle D, I, W, f, \textsc{ok}\rangle$; D the domain, I_w a first-order structure at each w. Assume a constant d' for each $d \in D$.

$$[\![Ft_1, \ldots, t_n]\!] = \big\{\{w' : (I_{w'}(t_1), \ldots, I_{w'}(t_n)) \in I_{w'}(F)\}\big\}$$

$$[\![\forall xAx]\!] = \{p \cap q \cap \cdots \;:\; p \in [\![Ad'_0]\!],\ q \in [\![Ad'_1]\!],\ \ldots\}$$

$$[\![\exists xAx]\!] = \bigcup_{d \in D} [\![Ad']\!]$$

Universal quantification = generalised conjunction (pairwise intersection).
Existential quantification = generalised disjunction (union).

Follows Ciardelli, Groenendijk & Roelofsen (2018:61). With (18) and (21) this predicts $\Diamond\exists xAx \models \forall x\Diamond Ax$ and $(\exists xAx) \mathrel{\Box\!\!\rightarrow} C \models \forall x(Ax \mathrel{\Box\!\!\rightarrow} C)$.

Flattening in the first-order case

(34a) If anyone ate John's sandwich, he'd be upset. $(\exists x Fx) \mathbin{\Box\!\!\rightarrow} Ga$

(34b) If anyone ate John's sandwich, Bob did. $(!\exists x Fx) \mathbin{\Box\!\!\rightarrow} Fb$

(34a) implies: for every salient person, if they ate it, John'd be upset.
(34b) does not.

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(34a) implies: for every salient person, if they ate it, John'd be upset.
(34b) does not.

And under probability, in the dessert scenario:

(35) What is the probability that Alice may eat any of the desserts?

- Answer 1: $\Pr(\Diamond !\exists x Fx)$ — however the die lands she may eat *some* dessert.
- Answer 0: $\Pr(\Diamond \exists x Fx)$ — she never has a *choice*.

Caveat (Embry 2014): not everything expressible with $\exists$ raises alternatives. *It snowed yesterday* $= \exists e(\ldots)$, but doesn't imply *if any snowing event had happened, I'd have gone skiing* (100 feet of snow: I stay home). Event quantification should be $!\exists$. The existential meaning must be *visible* to the alternative mechanism.

Deontic gaps: Barker's objection

> *"There is a difference between weak permission, which is the absence of prohibition, and strong permission [...] which is the assertion that some action is explicitly ok."* — *Barker (2010:12)*

On the **strong** theory, with $F(A) \equiv A \mathrel{\Box\!\!\rightarrow} \neg\textsc{ok}$:
$A \mathrel{\Box\!\!\rightarrow} \textsc{ok}$ and $A \mathrel{\Box\!\!\rightarrow} \neg\textsc{ok}$ can *both* fail. Room for gaps. ✓

On the **weak** theory, it looks as if everything is either permitted or forbidden. Gaps impossible?

Deontic gaps: the fix

Relocate the gap from the *semantics of permission* to the *notion of* OK.

Three classes of worlds — the good, the bad, and the gappy:

- OK: explicitly compatible with the rules
- BAD: explicitly incompatible
- neither: undecided

(36)

$$P(A) \equiv A \Diamond\!\!\rightarrow \text{OK}$$
$$O(A) \equiv \neg A \Box\!\!\rightarrow \text{BAD}$$
$$F(A) \equiv A \Box\!\!\rightarrow \text{BAD}$$

Whenever the selected A-worlds contain no OK-world but some gappy world, A is neither permitted nor forbidden. Gaps and weak permission coexist.

Two footnotes on gaps

1. Are gaps merely epistemic?
Arguably not. If a justice system has never ruled on whether some behaviour is legal, there is arguably *no fact of the matter* — not merely that we haven't discovered it (Kress 1989, Madry 1999).

2. Recovering the equivalences.
The tripartite division risks losing $\neg P(A) \equiv O(\neg A) \equiv F(A)$. Recover them where wanted with homogeneity: $P(A)$ and $F(A)$ presuppose every selected A-world is OK or BAD; $O(A)$ presupposes the same for $\neg A$-worlds.

3. Barker's apple-or-pear case is not a gap.
Barker: *You may eat an apple or a pear* "neither permits nor forbids eating both". But that's an **implication gap**, not a **deontic gap** — *You may dance* neither permits nor forbids singing, which is perfectly ordinary. Getting a deontic gap requires the extra step that the sentence is neither permitted nor forbidden.

An open problem: double negation

In alternative semantics, $\neg\neg(A \vee B) \neq A \vee B$: the first has one alternative, the second two.

(37a) If Sue hired Alice or Bob, she would hire Alice. $!(A \vee B) \mathrel{\Box\!\!\rightarrow} A$

(37b) Sue must hire neither Alice nor Bob, but she may hire Bob.
$\Box\neg(A \vee B) \wedge \Diamond B$

Intuitively inconsistent. But the account predicts consistency:

- $\Box\neg(A \vee B) \equiv \neg\neg(A \vee B) \mathrel{\Box\!\!\rightarrow} \neg\textsc{ok} \equiv\ !(A \vee B) \mathrel{\Box\!\!\rightarrow} \neg\textsc{ok}$
- (37a): every selected $!(A \vee B)$-world is an A-world
- So we only learn A-worlds aren't OK. B-worlds may still be. Hence $\Diamond B$.

Ad hoc patch: keep $[\![\neg A]\!] = \{\bigcup[\![A]\!]\}$ when A isn't a negation, and set $[\![\neg\neg A]\!] = [\![A]\!]$. Cf. analogous moves for anaphora (Gotham 2019, Hofmann 2022, Mandelkern 2022). More elegant solutions welcome.

Part 7

Comparisons

Six strong theories, six conditionals

Strong conditional theory	**Interpretation of "if A, OK"**
Anderson (1967)	Relevant implication (Anderson & Belnap 1962)
von Wright (1968)	A is a "sufficient condition" for OK
Hilpinen (1982), Nute (1985)	Variably strict conditional (Stalnaker 1968, Lewis 1973)
Asher & Bonevac (2005)	Defeasible conditional of non-monotonic logic
Lokhorst (1997), Barker (2010)	Conditional of linear logic (Girard 1987, Avron 1988)
Li (2025)	Dynamic strict conditional (von Fintel 2001b, Gillies 2007)

Mixed cases challenge every one of them.

Relevance logic (Anderson 1967)

"A is permitted" iff $A \to \textsc{ok}$, with $\to$ relevant implication in R.

Relevant implication in R is transitive, hence satisfies antecedent strengthening:

$$A \to B,\ B \to \textsc{ok} \vdash_R A \to \textsc{ok}$$

Plausibly *the switches are both down* relevantly implies *the switches are in the same position*: the first guarantees the second, and they are clearly about the same subject matter.

Predicts: "the switches are allowed to agree" $\Rightarrow$ "they are allowed to both be down". Wrong.

Strict and variably strict

von Wright (1968), Li (2025): $\Diamond A \equiv \Box(A \supset \textsc{ok})$.
Hilpinen (1982), Nute (1985): the closest A-worlds are OK.

These use the *same* semantic components as natural-language conditionals, so they predict permission statements to pattern with them. Mixed cases show they don't.

Strict and variably strict

von Wright (1968), Li (2025): $\Diamond A \equiv \Box(A \supset \textsc{ok})$.
Hilpinen (1982), Nute (1985): the closest A-worlds are OK.

These use the *same* semantic components as natural-language conditionals, so they predict permission statements to pattern with them. Mixed cases show they don't.

Escape route? Disassociate them — two selection functions:

(38a) $A \mathrel{\Box\!\!\rightarrow} C$ true at w iff $f(|A|, w) \subseteq |C|$

(38b) $\Diamond A$ true at w iff $f'(|A|, w) \subseteq \textsc{ok}_w$

For this to work, f' must be sensitive to the deontic ideals: if the rules require A up, f' must select only ↑↑; if they require A down, only ↓↓.

Two ways to make f' rule-sensitive; both fail

Simple: restrict to OK-worlds.

$$f'(p,w) = f(p,w) \cap \textsc{ok}_w$$

Trivialises permission: $f(p,w) \cap \textsc{ok}_w \subseteq \textsc{ok}_w$ always. Everything is permitted.

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Sophisticated: *prioritise* OK-worlds.

Modify the similarity order: $u <'_w v$ iff either u is OK and v isn't, or they have the same deontic status and $u <_w v$.

Now consider a world where taking an apple is permitted, taking a pear forbidden. The new order ranks apple-worlds closer than pear-worlds.

Predicts "You may take an apple or a pear" true.

Wrong — and it sacrifices the strong theory's whole selling point, the reduction of free choice to SDA.

Non-monotonic logic (Asher & Bonevac 2005)

$f(A, w)$ returns "the most **normal** A-worlds relative to w".

> *"If A, then normally B is true at w iff B is true in all normal A-worlds, that is, all A-worlds in which conditions are suitable for assessing (relative to w) what happens when A."* *(2005:309)*

But normality isn't what's at play. Take the Quebec case:

- Alice and Bob are intuitively permitted to talk at work.
- There is no suggestion that doing so *in French* is more normal — quite the opposite. They habitually speak English.
- So the "normal" worlds are English-speaking worlds, which are forbidden.

Predicts they are forbidden from talking to each other at work. Wrong.

Linear logic (Lokhorst 1997, Barker 2010)

$\Diamond A \equiv A \multimap \text{OK}$, with $\multimap$ linear implication.

Linear implication is a species of material implication:
$A \multimap B \equiv A^\perp \parr B \equiv (A \otimes B^\perp)^\perp$,
where $\perp$ is linear negation, $\parr$ multiplicative disjunction, $\otimes$ multiplicative conjunction.

So it inherits classical properties. As Barker (2010:13) notes, A occurs in a downward-entailing position — crucial to his derivations.

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But mixed cases show permission is *not* downward-entailing:

- *The switches are both down* $\models$ *the switches agree*,
  but *allowed to agree* $\not\models$ *allowed to both be down*.
- *A&B talk at work in English* $\models$ *A&B talk at work*,
  but *permitted to talk at work* $\not\models$ *permitted to talk in English*.

Formally: with $\mathcal{A} = \wp(\{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\})$, $\textsc{ok} = \{\uparrow\uparrow, \uparrow\downarrow\}$, "agree" $= \{\uparrow\uparrow, \downarrow\downarrow\}$ and "both down" $= \{\downarrow\downarrow\}$, the two permission statements come out equivalent (both true at $\{\uparrow\uparrow, \uparrow\downarrow, \downarrow\downarrow\}$).

An important distinction: coarseness $\neq$ mixed cases

Kamp (1973:63) on *Michael may go to the beach*:

> *Surely not every way of making the statement true. He is not to go by taxi; he is not to go to that section of the beach where there is no swimming guard. He is not to put on the coat which his mother bought him last month [...]. All these ways of going to the beach remain proscribed.*

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The strong theory has a good answer here. The selection function is *itself* coarse (Embry 2014): supposing "if I feed this man", we don't consider poison-worlds. Indeed *If I feed this man, I won't feed him poison* is intuitively true.

Takeaway

The difference: in coarseness cases, the permission statement and the conditional pattern *the same way*. In mixed cases, they *come apart*. That contrast is what the strong theory cannot explain.

Alternatives from negation? (Schulz 2018; Romoli et al. 2022)

They propose $\neg(A \wedge B)$ has three alternatives: $A \wedge \neg B$, $\neg A \wedge B$, $\neg A \wedge \neg B$.

Paired with our theory, $\neg\Box(A \wedge B)$ would entail all of $\Diamond(A \wedge \neg B)$, $\Diamond(\neg A \wedge B)$, $\Diamond(\neg A \wedge \neg B)$. Far too strong.

(39) You don't have to read both *Beloved* and *Song of Solomon*, but you do have to read at least one.

$$\neg\Box(A \wedge B) \wedge \Box(A \vee B)$$

Perfectly consistent

If $\neg\Box(A \wedge B)$ implied $\Diamond(\neg A \wedge \neg B)$, this should be as bad as *You may have ice cream or cake, but you may not have cake*. It isn't.

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$$\neg\Box(A \wedge B) \wedge \Box(A \vee B)$$ Perfectly consistent

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Epistemics behave the same:

(40a) Alice might not have read both books, but she read at least one.

(40b) Out of the two, Alice must have read *Beloved*, but she might not have read both.

Both fine — far better than *Alice might have read Beloved or Song of Solomon, but she didn't read Beloved*.

Part 8

Conclusion

The argument in five steps

1. Many authors think modals and conditionals draw on the same semantic source. Tradition: *A is permitted* $\approx$ *if A, the ideals would be met* — strong.

2. **Mixed cases** refute this. For *A* to be permitted it suffices that supposing *A* returns *some* OK world. And the strong analysis has independent counterexamples (tickets, De Morgan).

3. Weak permission runs straight into **free choice**.

4. **Alternatives** rescue it: permission is weak within each alternative, strong across them. Free choice *and* dual prohibition follow, with homogeneity derived rather than stipulated.

5. **Flattening** gives an ambiguity account handling SDA counterexamples and probability. Generalises to arbitrary modal flavours, to *any*, and to deontic gaps.

Open question: wide scope free choice

The inference from $\Diamond A \vee \Diamond B$ to $\Diamond A \wedge \Diamond B$ (Zimmermann 2000, Geurts 2005, Schulz 2005, Aloni 2022):

(41) You may go to Brixton or you may go to Victoria.
  a. $\rightsquigarrow$ You may go to Brixton.
  b. $\rightsquigarrow$ You may go to Victoria.

(42) He might be in Brixton or he might be in Victoria.

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If the reduction is right, conditionals should do the same. Remarkably, they do (Wiggins & Edgington 1997, Geurts 2005, Khoo 2021a, McHugh 2025):

(43) If you had gone to Brixton, you would have been fine, or if you had gone to Victoria, you would have been fine.
  a. $\rightsquigarrow$ If you had gone to Brixton, you would have been fine.
  b. $\rightsquigarrow$ If you had gone to Victoria, you would have been fine.

The present theory does not capture this. The parallel between modals and conditionals may run deeper than we have modelled.

Your turn

Discussion — pick one.

1. The three pragmatic diagnostics (QUD-sensitivity, cancellability, metalinguistic negation) are supposed to show the strong inferences are implicatures. How convincing is each? Can you think of a fourth test?

2. Generalisation (27) is stipulated. What deeper principle might it follow from? (Hint: what do a consequent that entails the antecedent and a probability operator have in common?)

3. The double negation problem (§7.4) got an admittedly ad hoc fix. Sketch a more principled alternative.

4. Can you extend the account to *wide scope* free choice? What would have to change?

Key references

- Alonso-Ovalle, L. (2006). *Disjunction in Alternative Semantics*. PhD thesis, UMass Amherst.
- Alonso-Ovalle, L. (2009). Counterfactuals, correlatives, and disjunction. *Linguistics and Philosophy* 32(2).
- Anderson, A. R. (1967). Some nasty problems in the formal logic of ethics. *Noûs* 1(4).
- Asher, N. & D. Bonevac (2005). Free choice permission is strong permission. *Synthese*.
- Barker, C. (2010). Free choice permission as resource-sensitive reasoning. *Semantics and Pragmatics* 3.
- Cariani, F. & S. Goldstein (2020). Conditional heresies. *Philosophy and Phenomenological Research*.
- Ciardelli, I., J. Groenendijk & F. Roelofsen (2018). *Inquisitive Semantics*. OUP.
- Hilpinen, R. (1982). Disjunctive permissions and conditionals with disjunctive antecedents.
- Kamp, H. (1973). Free choice permission. *Proceedings of the Aristotelian Society* 74.
- Kratzer, A. (1977). What "must" and "can" must and can mean. *Linguistics and Philosophy* 1.
- McKay, T. & P. van Inwagen (1977). Counterfactuals with disjunctive antecedents. *Philosophical Studies* 31.
- Prior, A. N. (1959). Escapism. In *Essays in Moral Philosophy*.
- Santorio, P. (2018). Alternatives and truthmakers in conditional semantics. *Journal of Philosophy* 115(10).
- Santorio, P. (2021). Path semantics for indicative conditionals. *Mind* 131(521).
- von Wright, G. H. (1968). *An Essay in Deontic Logic and the General Theory of Action*.
- Zimmermann, T. E. (2000). Free choice disjunction and epistemic possibility. *Natural Language Semantics* 8.

Thank you

Permission is weak within alternatives,
strong across them.

Appendix

Backup slides

Backup: all semantic clauses in one place

Alternative semantics (propositional)

$$[\![p]\!] = \{V(p)\} \qquad [\![A \wedge B]\!] = \{p \cap q : p \in [\![A]\!], q \in [\![B]\!]\}$$

$$[\![\neg A]\!] = \{\overline{\bigcup [\![A]\!]}\} \qquad [\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$$

$$[\![!A]\!] = \{\bigcup [\![A]\!]\}$$

Conditionals (defined only if all alternatives agree in status)

$$[\![A \mathrel{\Box\!\!\rightarrow} C]\!] = \big\{\{w' : \forall p \in [\![A]\!],\ f(p, w') \subseteq |C|\}\big\}$$

$$[\![A \mathrel{\Diamond\!\!\rightarrow} C]\!] = \big\{\{w' : \forall p \in [\![A]\!],\ f(p, w') \cap |C| \neq \emptyset\}\big\}$$

Modals

$$[\![\Box A]\!] = \big\{\{w' : \forall p \in [\![\neg A]\!],\ f(p, w') \subseteq \mathrm{OK}_{w'}\}\big\}$$

$$[\![\Diamond A]\!] = \big\{\{w' : \forall p \in [\![A]\!],\ f(p, w') \cap \mathrm{OK}_{w'} \neq \emptyset\}\big\}$$

Quantifiers

$$[\![\forall x A x]\!] = \{p \cap q \cap \cdots : p \in [\![A d_0']\!], q \in [\![A d_1']\!], \ldots\} \qquad [\![\exists x A x]\!] = \bigcup_{d \in D} [\![A d']\!]$$

Backup: Kratzer's OK

Context supplies a modal base b and ordering source g, both $W \to \wp(\wp(W))$.

$$w' \leq_{g(w)} w'' \text{ iff } \forall p \in g(w):\ w'' \in p \Rightarrow w' \in p$$
$$w' <_{g(w)} w'' \text{ iff } w' \leq_{g(w)} w'' \text{ but not } w'' \leq_{g(w)} w'$$
$$\textsc{best}(b, g, w) = \{w' \in \bigcap b(w) : \neg\exists w'' \in \bigcap b(w):\ w'' <_{g(w)} w'\}$$

Set $\textsc{ok}_w = \textsc{best}(b, g, w)$.

Note: the tripartite OK/BAD/gap division of §7.3 goes beyond this, which only divides worlds in two. Deriving a tripartite division inside a Kratzerian framework is an open question.

Backup: alternative vs. inquisitive semantics

Nothing today hinges on the choice. One could equally use inquisitive semantics (Ciardelli 2016; Ciardelli, Groenendijk & Roelofsen 2018).

The one divergence: **Hurford disjunctions**, $A \vee (A \wedge B)$, where one disjunct entails the other. The two frameworks assign different alternatives (Ciardelli & Roelofsen 2017).

Hurford's constraint (1974) predicts these to be odd anyway, unless the weaker disjunct is exhaustified: $A \vee (A \wedge B) \to (A \wedge \neg B) \vee (A \wedge B)$. Since they are independently odd, we set them aside.

Backup: why the subjunctive, in more detail

The security guard, required to stay at his post, can felicitously say:

- *I am required to be at my post right now.*
- *I'm not allowed to be elsewhere right now.*

Subjunctive reduction: *If I were not at my post right now, the rules would be broken.* Fine.

Indicative reduction: *If I am not at my post right now, the rules are/will be broken.* Odd.

Why? Indicatives are restricted to worlds compatible with the common ground (Stalnaker 1975). Everyone knows he's at his post.

The "right now" matters: it blocks a generic reading. Without it, *If I am not at my post, the rules will be broken* is acceptable even when he is at his post.

Backup: solutions to the exercises (sketch)

Part 4 exercise

1. $[\![p \vee q]\!] = \{|p|, |q|\}$ — two
2. $[\![\neg(p \vee q)]\!] = \{\overline{|p| \cup |q|}\}$ — one
3. $[\![\neg p \wedge \neg q]\!] = \{\overline{|p|} \cap \overline{|q|}\}$ — one (same as 2)
4. $[\![(p \vee q) \wedge r]\!] = \{|p| \cap |r|,\ |q| \cap |r|\}$ — two
5. $[\![!(p \vee q) \wedge r]\!] = \{(|p| \cup |q|) \cap |r|\}$ — one
6. $[\![\neg\neg(p \vee q)]\!] = \{|p| \cup |q|\}$ — one (the trap: not equivalent to 1 in alternative semantics; see §7.4)

Free choice arises for 1 and 4.

Part 5 exercise

1. Consequent entails antecedent ⇒ flattening available ⇒ can be true.
2. Neither condition ⇒ no flattening ⇒ SDA ⇒ entails both simplifications.
3. Probability operator ⇒ both readings available.
4. Unembedded ⇒ unflattened ⇒ free choice.
5. Probability operator ⇒ ambiguous: high (flattened) vs. possibly zero (unflattened).